

The Quantum Steering Ellipsoid

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The quantum ellipsoid of a two qubit state is the set of Bloch vectors that Bob can collapse Alice's qubit to when he implements all possible measurements on his qubit. We provide an elementary construction of the ellipsoid for arbitrary states, and explain how this geometric representation can be made faithful. The representation provides a range of new results, and uncovers new features, such as the existence of 'incomplete steering' in separable states. We show that entanglement can be analysed in terms of three geometric features of the ellipsoid, and prove that a state is separable if and only if it obeys a 'nested tetrahedron' condition. We provide a volume formula for the ellipsoid in terms of the state density matrix, which identifies exactly when steering is fully three dimensional, and identify this as a new feature of correlations, intermediate between discord and entanglement.

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The Bloch sphere provides a remarkably simple representation for the state space of the most primitive quantum unit - the single qubit system - and results in geometric intuitions that are invaluable in countless fundamental information-processing scenarios. Unfortunately, such a direct three-dimensional depiction of the state space is impossible for any quantum system of larger dimension.

While the qubit is the primitive unit of quantum information, the two qubit system constitutes the primitive unit for the theory of bipartite quantum correlations. However, the two qubit state space is already 15-dimensional and possesses a surprising amount of structure and complexity. As such, it is a highly non-trivial challenge to both faithfully represent its states and to acquire natural intuitions for their properties [1–3].

The phenomenon of steering in quantum states was first uncovered by Schrödinger [4] (and subsequently rediscovered by others [5–7]), who realised that measurements on B on the pure entangled state $|\psi\rangle_{AB}$ could be used to “steer” the state at A into all different convex decompositions of the reduced state ρ_A ; in particular for rank-one POVM measurements at B we have that A is steered into ensembles of pure states. More significantly, given any ensemble decomposition of ρ_A into either pure or mixed states there always exists a POVM measurement at B that generates that ensemble. In this regard we say that the steering at A for the pure state $|\psi\rangle_{AB}$ is *complete* within the Bloch sphere of A . More generally, for a 2-qubit mixed state ρ_{AB} it is known [8] that the convex set of states that A can be steered to is instead an ellipsoid \mathcal{E}_A that contains ρ_A , as in figure 1.

In this Letter we show that all correlation features of a two qubit state are encoded in its steering ellipsoid and local Bloch vectors, and argue that this ellipsoid representation for two qubits provides the natural generalization of the Bloch sphere picture, in that it gives a simple geometric representation of an arbitrary two qubit state ρ in three dimensions, which makes the key properties of the state manifest in simple geometric terms.

By adopting the ellipsoid representation we are lead to a range of novel results and fresh insights into quan-

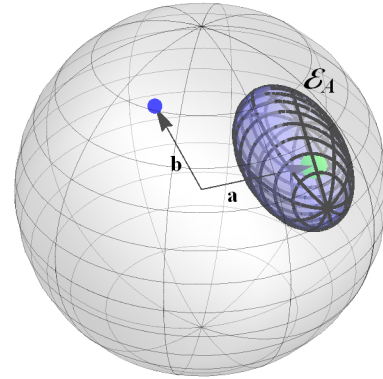


FIG. 1: **Ellipsoid representation of a two-qubit state.** For any two-qubit state ρ_{AB} , the set of states to which Bob can steer Alice forms an ellipsoid \mathcal{E}_A in Alice's Bloch sphere, containing her Bloch vector \mathbf{a} . The inclusion of Bob's Bloch vector \mathbf{b} determines ρ_{AB} up to a choice of basis for Bob, which can be fixed by indicating the orientation of \mathcal{E}_A .

tum correlations for both entangled and separable states. Firstly, we provide an alternative construction to the steering ellipsoid \mathcal{E}_A to that in [9], which applies even when \mathcal{E}_A is degenerate. We then provide a reconstruction of a state ρ_{AB} from its geometric data to explore the faithfulness of the representation. This analysis reveals a new phenomenon of incomplete steering for certain separable quantum states, in which some decompositions of ρ_A within of the steering ellipsoid \mathcal{E}_A are inaccessible.

The representation then allows us to decompose entanglement into simple geometric features, and show how it depends only on (a) the spatial orientation of the ellipsoid, (b) its distance from the origin and (c) its size. The representation also leads us to establish the surprising *nested tetrahedron condition*: a state ρ_{AB} is separable if and only if its ellipsoid fits inside a tetrahedron that itself fits inside the Bloch sphere. We then study the minimal number of product states in the ensemble of a separable state, which we show is determined solely by the dimension of \mathcal{E}_A . We then develop a useful volume for-

mula that identifies exactly when steering is fully three-dimensional. Finally the work suggests a new feature of quantum correlations, called *obesity*, which is neither discord nor separability nor entanglement. We observe that quantum discord arises from a combination of both the obesity of the state and the orientation of its ellipsoid.

Beyond these new results, we also feel that the method of compactly depicting any two qubit state ρ_{AB} in three dimensions, via a single ellipsoid and two local Bloch vectors, should be of interest to a range of researchers in both the theoretical and experimental quantum sciences.

The Pauli basis. Let σ_0 denote the 2×2 identity matrix and σ_i , $i = 1, 2, 3$ be the usual Pauli matrices. Any single-qubit Hermitian operator E can be written $E = \frac{1}{2} \sum_{\mu=0}^3 X_\mu \sigma_\mu$, where the $X_\mu = \text{tr}(E \sigma_\mu)$ are real. E is positive iff $X_0 \geq 0$ and $X_0^2 \geq \sum_{i=1}^3 X_i^2$. With X viewed as a 4-vector, this identifies the set of positive operators with the usual forward light cone in Minkowski space. In the case of a qubit state, $X_0 = 1$ and the remaining components form the Bloch vector.

Similarly, we can write a two-qubit state as $\rho = \frac{1}{4} \sum_{\mu, \nu=0}^3 \Theta_{\mu\nu} \sigma_\mu \otimes \sigma_\nu$. The components of the 4×4 real matrix Θ are $\Theta_{\mu\nu} = \text{tr}(\rho \sigma_\mu \otimes \sigma_\nu)$. As a block matrix we have $\Theta = \begin{pmatrix} 1 & \mathbf{b}^T \\ \mathbf{a} & T \end{pmatrix}$, where \mathbf{a}, \mathbf{b} are the Bloch vectors of the reduced states ρ_A and ρ_B of ρ and T is a 3×3 matrix encoding correlations [2]. Now if Bob does a measurement and obtains a POVM outcome E , then Bob steers Alice to the state proportional to $\text{tr}_B(\rho(\sigma_0 \otimes E))$, with the latter being given by $\frac{1}{2} \Theta X$ in the Pauli basis.

The matrix Θ transforms, up to a normalization, under SLOCC operations $\rho' = S_A \otimes S_B \rho (S_A \otimes S_B)^\dagger$, where S_A, S_B are invertible complex matrices, as $\Theta' = \Lambda_A \Theta \Lambda_B^T$ [9] where Λ_A, Λ_B are proper orthochronous Lorentz transformations (see the appendices).

In the case of a SLOCC operation affecting only Bob ($\Theta' = \Theta \Lambda_B$), the set of states Bob can steer Alice to is unaffected: X is in the forward light cone if and only if $X' = \Lambda_B X$ is, and $\Theta' X = \Theta X'$.

Construction of the quantum steering ellipsoid for a given state ρ . Steering ellipsoids have been studied for the representation of SLOCC transformations [8], and more recently in relation to quantum discord [10].

The steering ellipsoid is easiest to understand for states with $\mathbf{b} = \mathbf{0}$. Suppose that Bob projects onto some pure state $X = \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix}$ with $v = 1$. Then

$$Y = \frac{1}{2} \Theta X = \frac{1}{2} \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{a} & T \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \mathbf{a} + T\mathbf{v} \end{pmatrix}, \quad (1)$$

showing that Bob will obtain that outcome with probability $1/2$ and Alice's Bloch vector will then be $\mathbf{a} + T\mathbf{v}$. The set of all states Alice can end up with is simply the unit sphere of possible \mathbf{v} , shrunk and rotated by T and translated by \mathbf{a} , i.e. an ellipsoid centred at \mathbf{a} with orientation and semiaxes given by the eigenvectors and eigenvalues of TT^T . The dimension of the ellipsoid is

$\text{rank}(T) = \text{rank}(\Theta) - 1$. Points inside the ellipsoid can be reached via convex combinations of projective measurements, and conversely any POVM element for Bob can be decomposed into a mixture of projections, thus giving a point within the ellipsoid.

Now consider a general state with $\mathbf{b} \neq \mathbf{0}$. If $b = 1$ then ρ is a product state, and so the steering ellipsoid is a single point. Hence assume that $b < 1$. Then the SLOCC operator $\mathbb{1} \otimes \rho_B^{-\frac{1}{2}}$ that corresponds to a Lorentz boost Λ_B by a 'velocity' \mathbf{b} , transforms ρ_B to the maximally mixed state. We refer to this filtered state $\tilde{\rho}$ as the *canonical state* on the SLOCC orbit $\mathcal{S}(\rho)$ (see appendix). As already noted, SLOCC operations on Bob do not affect Alice's steering ellipsoid. Therefore we can find the parameters of an arbitrary state's steering ellipsoid by boosting Θ by Λ_B and then reading off the ellipsoid parameters. This gives (see appendix) a steering ellipsoid centred at $\mathbf{c}_A = \frac{\mathbf{a} - T\mathbf{b}}{1 - b^2}$, and orientation and semiaxes lengths $s_i = \sqrt{q_i}$ given by the eigenvectors and eigenvalues q_i of the ellipsoid matrix

$$Q_A = \frac{1}{1 - b^2} (T - \mathbf{a}\mathbf{b}^T) \left(\mathbb{1} + \frac{\mathbf{b}\mathbf{b}^T}{1 - b^2} \right) (T^T - \mathbf{b}\mathbf{a}^T). \quad (2)$$

This together with \mathbf{c}_A , specify the ellipsoid \mathcal{E}_A .

To obtain the ellipsoid at B, \mathcal{E}_B , the roles of A and B are reversed, which amounts to a transposition of the matrix Θ . In terms of its components, this reads as $\mathbf{b} \rightarrow \mathbf{a}, \mathbf{a} \rightarrow \mathbf{b}, T \rightarrow T^T$, and thus \mathcal{E}_A and \mathcal{E}_B always have the same dimensionality, $\text{rank}(\Theta) - 1$. This completes the construction of the geometric data $(\mathcal{E}_A, \mathbf{a}, \mathbf{b})$ for a given state ρ . We next describe the reverse direction, obtaining ρ_{AB} from an ellipsoid \mathcal{E}_A and the vectors \mathbf{a} and \mathbf{b} [8].

Faithfulness of the representation: the reconstruction of ρ from geometric data. Since we are given \mathbf{a} and \mathbf{b} , all that remains is to obtain the correlation matrix T as a function of $(Q_A, \mathbf{c}_A, \mathbf{a}, \mathbf{b})$. We find that the components of the matrix are given by

$$T_{ij} = (c_A)_i b_j + \sum_{k=1}^3 (\sqrt{Q_A} M)_{ik} \text{tr}(\sqrt{\rho_B} \sigma_k \sqrt{\rho_B} \sigma_j) \quad (3)$$

where M is an orthogonal matrix, obtained from $M\mathbf{b} = (\sqrt{Q_A})^{-1}(\mathbf{a} - \mathbf{c}_A)$. This specifies M up to an orthogonal matrix $O_B \mathbf{b} = \mathbf{b}$, with $O_B \in O(3)$, which corresponds to the choice of bases at B. This choice of basis can be simply encoded, for example, by providing two contour lines on \mathcal{E}_A . The derivation is provided in the appendices.

A theorem on "complete" and "incomplete" steering. The steering ellipsoid specifies for which states there is at least one measurement outcome for Bob that steers Alice to it. A more subtle question is for which decompositions of Alice's reduced state ρ_A is there a measurement for Bob that steers to the decomposition. Clearly a necessary condition is that all of the states in the decomposition must be in \mathcal{E}_A , surprisingly however it turns out that this is not sufficient. The situation is captured by the following result.

Consider some non-product two-qubit state Θ with ellipsoids \mathcal{E}_A and \mathcal{E}_B . The following are equivalent:

1. (*Complete steering*) For any convex decomposition of \mathbf{a} into states in \mathcal{E}_A or on its surface, there exists a POVM for Bob that steers Alice to it.
2. Alice's Bloch vector \mathbf{a} lies on the surface of \mathcal{E}_A scaled down by b .
3. The affine span of \mathcal{E}_B contains the maximally mixed state.

The proof is in the appendices. In particular, these conditions hold for all non-degenerate ellipsoids (which includes all entangled states) as well as all states where $\mathbf{b} = \mathbf{0}$. An example of a state where the above conditions fail is the state $\rho = \frac{1}{2}(|00\rangle\langle 00| + |1+\rangle\langle 1+|)$.

The three geometric contributions to entanglement. The Peres-Horodecki criterion [11, 12] asserts that a two-qubit state ρ_e is entangled if and only if $\rho_e^{T_B}$ has a negative eigenvalue. It was shown in [9] that at most one eigenvalue of $\rho_e^{T_B}$ can be negative, furthermore in [13] it is demonstrated that $\rho_e^{T_B}$ is full rank for all entangled states. It follows that $\det \rho_e^{T_B} < 0$ is a necessary and sufficient condition for entanglement.

Suppose ρ is entangled, then any state in its SLOCC orbit $\mathcal{S}(\rho)$ is also entangled [9] and in particular, the canonical state $\tilde{\rho} \in \mathcal{S}(\rho)$. Therefore, we can use $\det(\rho^{T_B}) < 0 \Leftrightarrow \det(\tilde{\rho}^{T_B}) < 0$ to show that a quantum steering ellipsoid corresponds to an entangled state if and only if

$$c^4 + c^2(1 - \text{tr}(Q) + \hat{\mathbf{n}}^T Q \hat{\mathbf{n}}) + h(Q) < 0 \quad (4)$$

where $\mathbf{c} = c\hat{\mathbf{n}}$ and $h(Q) = 1 - 8 \det \sqrt{Q} + 2 \text{tr}(Q^2) - (\text{tr}(Q))^2 - 2 \text{tr}(Q)$. We have dropped the A, B labels for the centre \mathbf{c} and Q because entanglement is a “symmetric” relation. This equation is manifestly invariant under global rotations of the ellipsoid, corresponding to local unitaries on the quantum state. It elucidates that entanglement correlations between the qubits manifest themselves in three geometric properties: (1) the distance of \mathcal{E}_A from the origin, (2) The dimensions of \mathcal{E}_A , given by singular values of Q and (3) the “skew” of \mathcal{E}_A , captured by the term $\hat{\mathbf{n}}^T Q \hat{\mathbf{n}}$, which depends on the alignment of the ellipsoid relative to the radial direction.

The nested tetrahedron condition. The condition for entanglement given by equation (4), provides a compact algebraic condition for non-separability and uncovers contributions from different geometric aspects. However, the steering setting allows us to go further, and capture the distinction between separable and non-separable states in a remarkably elegant and intuitive way. The result is a single, geometric criterion for separability:

A two qubit state ρ is separable if and only if its steering ellipsoid \mathcal{E}_A fits inside a tetrahedron that fits inside the Bloch sphere.

Dim.	Type	Alice's ellipsoid	Bob's ellipsoid
3 (Obese)	Entangled		
3 (Obese)	Separable		
2 (Pancake)	Incomplete		
2 (Pancake)	Complete		
1 (Needle)	Incomplete		

TABLE I: **The classes of steering ellipsoids.** The green dot is the reduced state. States with \mathcal{E}_A being 3-dimensional have obesity and can be either entangled or separable. The state ρ is separable if and only if \mathcal{E}_A fits inside a tetrahedron inside the Bloch sphere of A . For separable states, the set \mathcal{E}_A can also be 2-dimensional (a steering pancake), or 1-dimensional (a steering needle), or trivially 0-dimensional (not shown). For these cases, steering is either “complete” if all ensemble decompositions of \mathbf{a} in \mathcal{E}_A are attainable (when the span of \mathcal{E}_B contains $\frac{1}{2}\mathbb{1}$), otherwise the steering is “incomplete”. Zero discord occurs only for radial steering needles.

To prove necessity, suppose Alice and Bob share a separable state $\rho = \sum_{i=1}^n p_i \alpha_i \otimes \beta_i$. Since we can always take $n \leq 4$ [14], the Bloch vectors of the α_i define a (possibly degenerate) tetrahedron \mathcal{T} within Alice's Bloch sphere. If Bob measures an outcome E then Alice is collapsed to $\sum_{i=1}^n \frac{p_i \text{tr}(E\alpha_i)}{\text{tr}(E\rho_B)} \alpha_i$. Hence her new Bloch vector will be a

convex combination of the Bloch vectors for the α_i — in other words her steering ellipsoid is contained in \mathcal{T} .

We prove in the appendix that the non-trivial converse holds: any ellipsoid that fits inside a tetrahedron that itself fits inside the Bloch sphere must arise from a separable state, and thus the nested tetrahedron condition is both necessary and sufficient for separability of the state.

Building further on this geometric understanding, we can prove the following result. Let ρ be separable and n minimal such that $\rho = \sum_{i=1}^n p_i \alpha_i \otimes \beta_i$. Then $n = \text{rank}(\Theta)$. Proof: since the Θ matrix for a product state has rank 1, it is clear that $n \geq \text{rank}(\Theta)$. Since any separable state can be written using 4 (pure) product states [14] we have $n \leq 4$. If $\text{rank}(\Theta) = 1$ then ρ is a product state and so $n = 1$. If $\text{rank}(\Theta) = 2$ then \mathcal{E}_A is a line segment and we can form a decomposition of ρ using the endpoints of its steering ellipsoid, giving $n = 2$. The case $\text{rank}(\Theta) = 3$ (where \mathcal{E}_A is an ellipse) is non-trivial, but is solved in the appendix by using the fact that any ellipse inside a tetrahedron inside the unit sphere also fits inside a triangle inside the unit sphere [15].

Volume of the ellipsoid. We can now arrive at a compact and useful expression for the volume of the steering ellipsoids, which allows a simple inference as to the type of steering a given state ρ provides. The volume of any ellipsoid is proportional to the product of its semiaxes $V = \frac{4\pi}{3} s_1 s_2 s_3$. Therefore the \mathcal{E}_A has volume $V_A = \frac{4\pi}{3} |\sqrt{\det Q_A}|$, which may be rewritten as $V_A = \frac{4\pi}{3} \frac{|\det \Theta|}{(1-b^2)^2}$. However, it turns out (see appendix) that $|\det \Theta| = 16|\det \rho - \det \rho^{T_B}|$, and so we obtain that

$$V_A = \frac{64\pi}{3} \frac{|\det \rho - \det \rho^{T_B}|}{(1-b^2)^2}. \quad (5)$$

The \mathcal{E}_B volume can be calculated from V_A via the simple relation $V_B = \frac{(1-b^2)^2}{(1-a^2)^2} V_A$, and thus if one side has non-zero volume then so too must the other. This volume is a witness of entanglement, in the sense any state with $V_A > V_\star = 4\pi/81$ must be entangled. This can be seen either from the nested tetrahedron theorem in the earlier section or from equation (4). With the former one simply constructs the largest tetrahedron possible in the Bloch sphere, while for the latter one observes that c must vanish, and then maximizes subject to $h(Q) = 0$. In both cases one obtains V_\star , namely the volume of the Werner state on the separable-entangled boundary.

Quantum discord and the ellipsoid. Quantum discord has received much attention as a measure of the quantumness of correlations (see [16] for details) in which zero discord for one party roughly corresponds to them possessing a non-disturbing projective measurement. In the appendix, we show that ρ has zero discord for A if and only if \mathcal{E}_A is a segment of a diameter, while ρ has zero discord for B if and only if \mathcal{E}_A is one-dimensional and $b = \frac{2|c_A - a|}{l_A}$, where l_A is the length of \mathcal{E}_A .

A new perspective on quantum correlations: obesity. Consider a game in which Bob must convince Alice that

the qubits they each hold possess more-than-classical correlations: he succeeds in his task if he can steer Alice to 3 states with linearly independent Bloch vectors. A state that is a resource for this game is called “obese”: it has an ellipsoid with finite volume. Obesity is neither the same as entanglement nor separability nor discord. All entangled states have $\det \rho^{T_B} < 0$ and so it follows from (F3) that all entangled states have non-zero volume and hence are obese. Equation (F3) also implies that the set of separable states divides into obese and non-obese (skinny) states. All zero discord states are skinny, however one can have \mathcal{E}_A being one-dimensional (steering needles) but the state having positive discord due to \mathcal{E}_A not being radially aligned. In this sense, quantum discord can be viewed as arising from a combination of an obesity component and a skew contribution.

It turns out that $V_A \propto |\det(T - \mathbf{a}\mathbf{b}^T)|$, and so an obese state must have $T - \mathbf{a}\mathbf{b}^T$ being full rank. Since $\mathbf{a}\mathbf{b}^T$ is rank-one we must have $\text{rank}(T) \geq 2$, however we cannot simply reduce the condition of non-zero obesity to full rank T , since there exist states with full rank T that have degenerate steering ellipsoids (ellipses) and so have zero volume, for example $\mathbf{a} = \mathbf{b} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ and $T = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. However, if we replace this T with $T = \text{diag}(\frac{1}{3}, \frac{1}{3}, 0)$ we obtain an obese state with $\text{rank}(T) = 2$.

To contrast entanglement, discord and obesity, we can consider a family of canonical states $\rho(\theta) = \frac{1}{4} \left(\mathbb{1} + \frac{1}{2} \sigma_z \otimes \mathbb{1} + \sum_{i,j} T_{ij}(\theta) \sigma_i \otimes \sigma_j \right)$ in which the skew of the ellipsoid smoothly varies via the elements of $T(\theta) = R_y(\theta) T R_y^T(\theta)$, with $R_y(\theta)$ being a rotation about the y axis by $\theta \in [0, \pi)$ and $T = \text{diag}(-\frac{9}{20}, -\frac{3}{10}, -\frac{3}{10})$. The ellipsoids have equal volumes and are centred at $\mathbf{c}_A = (0, 0, \frac{1}{2})^T$. This family of states illustrate an opposing behavior of the discord and concurrence as a function of skew, see Fig. 2. The entanglement favors an orientation in which the longest ellipsoid semiaxis is aligned radially, while discord is maximised when this axis is orthogonal to the radial direction [22].

Conclusion. The quantum steering ellipsoid provides a faithful representation of any two qubit state and a natural geometric classification of states (as in Table 1). It provides clear and intuitive understanding into the usual key aspects of two qubit states, uncovers surprising new features (such as the nested tetrahedron condition, skew & obesity, and incomplete steering) while prompting novel questions, such as: can we use (4) to define a class of “least-classical” separable states for fixed $(\mathbf{a}, \mathbf{b}, \mathbf{c})$? Can we use the nested tetrahedron condition to provide a simple construction for the best separable approximation [17] for a state ρ ? What is the characterisation of steering ellipsoids for real states? This would be particularly useful experimentally, acting as a new type of tomography for two qubits.

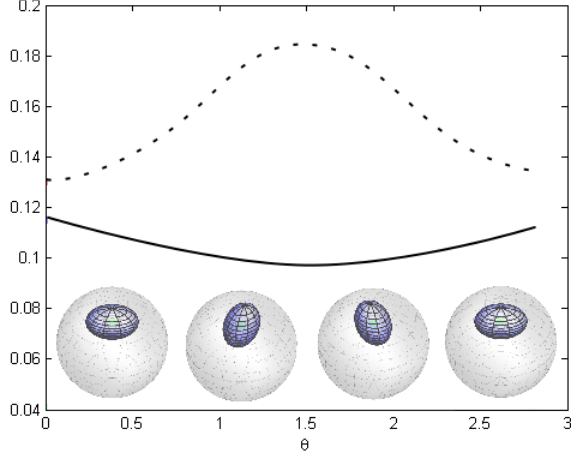


FIG. 2: Discord (solid) and concurrence (dotted) of the states $\rho(\theta)$ as a function of the orientation of the ellipsoid. Entanglement is maximized when the major axis is radial.

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This following appendices contain the details for some of the statements and theorems that are presented in the main article.

Appendix A: The steering ellipsoid for a general two qubit state

Previous approaches to representing two qubit states have included partitioning the set of all two qubit states into SLOCC (Stochastic Local Operations and Classical Communication) equivalence classes [1], which results in a three dimensional representation of a state ρ through its SLOCC orbit, defined as

$$\mathcal{S}(\rho) := \left\{ \frac{S_A \otimes S_B \rho (S_A \otimes S_B)^\dagger}{\text{tr}(S_A \otimes S_B \rho (S_A \otimes S_B)^\dagger)} : S_A, S_B \in \text{GL}(2, \mathbb{C}) \right\}$$

with $\text{GL}(2, \mathbb{C})$ the group of invertible, complex 2×2 matrices. However replacing ρ with its SLOCC orbit is far from faithful, and amounts to a highly coarse-grained representation of the state that erases much of its detail. Another approach is to start with a Pauli basis expansion of ρ , which can be converted via a state-dependent choice of local unitaries on both qubits, to a representation involving three spatial vectors [18]. Again, this is still not a faithful representation, and more importantly it is extremely difficult to develop any intuition for what the vector representing correlations actually means.

In this section we provide the details for constructing the steering ellipsoid representation of an arbitrary two qubit quantum state.

Consider a two-qubit state

$$\Theta = \begin{pmatrix} 1 & \mathbf{b}^T \\ \mathbf{a} & T \end{pmatrix}. \quad (\text{A1})$$

The matrix Θ transforms, up to a normalization, under SLOCC operations $\rho' = S_A \otimes S_B \rho (S_A \otimes S_B)^\dagger$ as $\Theta' = \Lambda_A \Theta \Lambda_B^T$ [9] where Λ_A, Λ_B are proper orthochronous Lorentz transformations given by $\Lambda_W = \Upsilon S_W \otimes S_W^* \Upsilon^\dagger / |\det S_W|$, $W \in \{A, B\}$ and

$$\Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A2})$$

In particular, local unitaries rotate the Pauli basis: $\mathbf{a} \rightarrow O_A \mathbf{a}$, $\mathbf{b} \rightarrow O_B \mathbf{b}$, $T \rightarrow O_A T O_B^T$, where $O_A, O_B \in \text{SO}(3)$.

If $b = 1$ then Θ must be a product state and no steering can occur, so assume otherwise. Define $\gamma = 1/\sqrt{1-b^2}$, and a new state

$$\Theta' = \gamma \Theta L_{\mathbf{b}}, \quad (\text{A3})$$

where $L_{\mathbf{b}}$ is a Lorentz boost by \mathbf{b} [19]:

$$L_{\mathbf{b}} = \begin{pmatrix} \gamma & -\gamma \mathbf{b}^T \\ -\gamma \mathbf{b} & \mathbb{1} + \frac{\gamma-1}{b^2} \mathbf{b} \mathbf{b}^T \end{pmatrix}. \quad (\text{A4})$$

The γ in (A3) ensures that Θ' is normalized: the top-left element is

$$\gamma(1 \quad \mathbf{b}^T) \begin{pmatrix} \gamma \\ -\gamma\mathbf{b} \end{pmatrix} = \gamma^2(1 - b^2) = 1. \quad (\text{A5})$$

The boost means Bob's reduced state is maximally mixed: the top-right block of Θ' is

$$\gamma(1 \quad \mathbf{b}^T) \begin{pmatrix} -\gamma\mathbf{b}^T \\ \mathbb{1} + \frac{\gamma-1}{b^2}\mathbf{b}\mathbf{b}^T \end{pmatrix} = \gamma(-\gamma\mathbf{b}^T + \mathbf{b}^T + (\gamma-1)\mathbf{b}^T) = \mathbf{0}^T. \quad (\text{A6})$$

By the above two observations we can write

$$\Theta' = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{a}' & T' \end{pmatrix}. \quad (\text{A7})$$

The set of states Bob can steer Alice to will be exactly the same for Θ and Θ' : $Y = \frac{1}{2}\Theta X \iff Y = \frac{1}{2}\Theta' X'$ where $X' = L_{-\mathbf{b}}X/\gamma$ and X corresponds to a positive operator iff X' does because $L_{-\mathbf{b}}/\gamma$ preserves the forward light cone.

What is that set of states? Writing $X = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$ we see that without loss of generality we can take $t = 1$ since the effect of multiplying X by a positive number is undone when normalizing Y . Hence the positivity condition becomes $x \leq 1$. If $x < 1$ then we could write X as a convex combination of ones with $x = 1$, so we restrict attention to the latter case. So any state \mathbf{y} that Alice can steer Bob to is a convex combination of states of the form $\mathbf{y} = \mathbf{a}' + T'\mathbf{x}$ with $x = 1$. But this defines a linear image of the unit sphere displaced by \mathbf{a}' , or in other words an ellipsoid centred at \mathbf{a}' .

Consider the singular value decomposition $T' = O_1 D O_2$. O_2 simply rotates the unit sphere and so can be ignored. D stretches the sphere and thus gives the lengths of the semi-axes of the resulting ellipsoid. O_1 rotates the semi-axes. We have $T'T'^T = O_1 D^2 O_1^T$ and so the lengths of the semi-axes can also be found by square rooting the eigenvalues of $T'T'^T$ whilst their directions can be found from its eigenvectors.

Combining (A3) with (A7) we find

$$\mathbf{a}' = \gamma(\mathbf{a} \quad T) \begin{pmatrix} \gamma \\ -\gamma\mathbf{b} \end{pmatrix} = \gamma^2(\mathbf{a} - T\mathbf{b}), \quad (\text{A8})$$

$$T' = \gamma(\mathbf{a} \quad T) \begin{pmatrix} -\gamma\mathbf{b}^T \mathbb{1} + \frac{\gamma-1}{b^2}\mathbf{b}\mathbf{b}^T \end{pmatrix} \quad (\text{A9})$$

$$= \gamma \left(-\gamma\mathbf{a}\mathbf{b}^T + T + \frac{\gamma-1}{b^2}T\mathbf{b}\mathbf{b}^T \right). \quad (\text{A10})$$

And so, after some algebra,

$$T'T'^T = \gamma^2(TT^T - \mathbf{a}\mathbf{a}^T) + \mathbf{a}'\mathbf{a}'^T. \quad (\text{A11})$$

Using $-\mathbf{a}\mathbf{b}^T(\mathbb{1} + \frac{\gamma-1}{b^2}\mathbf{b}\mathbf{b}^T) = -\gamma\mathbf{a}\mathbf{b}^T$ we can also write

$$T' = \gamma(T - \mathbf{a}\mathbf{b}^T) \left(\mathbb{1} + \frac{\gamma-1}{b^2}\mathbf{b}\mathbf{b}^T \right). \quad (\text{A12})$$

leading to the form in the main text:

$$T'T'^T = \gamma^2(T - \mathbf{a}\mathbf{b}^T)(\mathbb{1} + \gamma^2\mathbf{b}\mathbf{b}^T)(T^T - \mathbf{b}\mathbf{a}^T) =: Q_A \quad (\text{A13})$$

Since $\gamma L_{\mathbf{b}}$ is invertible we have $\text{rank}(\Theta') = \text{rank}(\Theta)$. By counting linearly independent columns in (A7) we have $\text{rank}(\Theta') = \text{rank}(T') + 1$ thus proving that the dimension of the ellipsoid is $\text{rank}(\Theta) - 1$.

Appendix B: Proof of the complete steering theorem

We present an extended formulation of the theorem:

Theorem Consider some non-product two-qubit state $\Theta = \begin{pmatrix} 1 & \mathbf{b}^T \\ \mathbf{a} & T \end{pmatrix}$ with ellipsoids \mathcal{E}_A and \mathcal{E}_B . The following are equivalent:

1. Complete steering: for all convex decompositions of \mathbf{a} into states in \mathcal{E}_A , there exists a POVM for Bob that steers Alice to it.
2. Any surface steering: there exists a convex decomposition of \mathbf{a} into states on the surface of \mathcal{E}_A with a POVM for Bob that steers Alice to it.
3. Alice's Bloch vector lies on the surface of her ellipsoid scaled down by b .
4. The affine span of \mathcal{E}_B contains the maximally mixed state.
5. $\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \in \text{range}(\Theta^T)$.
6. $\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \in \ker(\Theta)^\perp$.

Proof. Preliminaries: Let $\gamma = 1/\sqrt{1-b^2}$ and $\Theta' = \gamma\Theta L_{\mathbf{b}}$. Then $\Theta = \Theta' \frac{L_{-\mathbf{b}}}{\gamma}$ and so 6 is equivalent to $\begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix} \in \ker(\Theta')^\perp$. If we write $\Theta' = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{a}' & T' \end{pmatrix}$ then we see that any vector in $\ker(\Theta')$ is of the form $\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}$ with $T'\mathbf{x} = \mathbf{0}$ and so 6 is equivalent to $\mathbf{b} \in \ker(T')^\perp$.

\mathcal{E}_A are the points that can be written $\mathbf{a}' + T'\mathbf{x}'$ where $x' \leq 1$. Therefore the surface of \mathcal{E}_A are the points that can be written $\mathbf{a}' + T'\mathbf{x}'$ where $x' = 1$ and $\mathbf{x}' \in \ker(T')^\perp$. Hence the scaled down surface is $\mathbf{a}' + T'\mathbf{x}'$ where $x' = b$ and $\mathbf{x}' \in \ker(T')^\perp$.

From $\Theta = \Theta' \frac{L_{-\mathbf{b}}}{\gamma}$ we calculate that $\mathbf{a} = \mathbf{a}' + T'\mathbf{b}$.

1 \implies 2: Trivial.

2 \implies 6: Let \mathbf{y}_i on the surface of \mathcal{E}_A form a convex decomposition $\sum_i p_i \mathbf{y}_i = \mathbf{a}$. Since they are on the surface, we have $\mathbf{y}_i = \mathbf{a}' + T'\mathbf{x}'_i$ where $x'_i = 1$ and $\mathbf{x}'_i \in \ker(T')^\perp$. Suppose we also have $\mathbf{y}_i = \mathbf{a}' + T'\mathbf{x}''_i$ with $x''_i \leq 1$. Then $\mathbf{x}'_i - \mathbf{x}''_i \in \ker(T')$, and the only

way that the difference between two vectors can be perpendicular to the longer one is if they are equal. Therefore $2p_i \begin{pmatrix} 1 \\ \mathbf{x}'_i \end{pmatrix}$ is the unique element of the forward light

cone that $\frac{1}{2}\Theta'$ maps to $p_i \begin{pmatrix} 1 \\ \mathbf{y}_i \end{pmatrix}$, and therefore $\gamma L_{\mathbf{b}}$ times these form the only possible POVM elements for Bob. But to be a valid POVM, they must sum to the identity $\begin{pmatrix} 2 \\ \mathbf{0} \end{pmatrix}$, i.e. $\sum_i 2p_i \begin{pmatrix} 1 \\ \mathbf{x}'_i \end{pmatrix} = \frac{L-\mathbf{b}}{\gamma} \begin{pmatrix} 2 \\ \mathbf{0} \end{pmatrix} = 2 \begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix}$. Since the $\mathbf{x}'_i \in \ker(T')^\perp$, this implies $\mathbf{b} \in \ker(T')^\perp$ which is equivalent to 6.

6 \implies 1: Let $\mathbf{y}_i \in \mathcal{E}_A$ form a convex decomposition $\sum_i p_i \mathbf{y}_i = \mathbf{a}$. Since $\mathbf{y}_i \in \mathcal{E}_A$ we have $\mathbf{y}_i = \mathbf{a}' + T' \mathbf{x}'_i$ where $x'_i \leq 1$. Write $\mathbf{x}'_i = \mathbf{k}_i + \mathbf{c}_i$ where $\mathbf{k}_i \in \ker(T')$ and $\mathbf{c}_i \in \ker(T')^\perp$. This implies $c_i \leq x'_i \leq 1$ and $\mathbf{y}_i = \mathbf{a}' + T' \mathbf{c}_i$. So $2p_i \begin{pmatrix} 1 \\ \mathbf{c}_i \end{pmatrix}$ are in the forward light cone and map to $p_i \begin{pmatrix} 1 \\ \mathbf{y}_i \end{pmatrix}$ under $\frac{1}{2}\Theta'$. Hence $\gamma L_{\mathbf{b}}$ times these are

in the forward line cone and map to $p_i \begin{pmatrix} 1 \\ \mathbf{y}_i \end{pmatrix}$ under $\frac{1}{2}\Theta$. Since $\sum_i p_i \mathbf{y}_i = \mathbf{a} = \mathbf{a}' + T' \mathbf{b}$ we have $T' \sum_i p_i \mathbf{c}_i = T' \mathbf{b}$. By construction $\mathbf{c}_i \in \ker(T')^\perp$ and by assumption $\mathbf{b} \in \ker(T')^\perp$, and so this implies $\sum_i p_i \mathbf{c}_i = \mathbf{b}$. Then $\sum_i \gamma L_{\mathbf{b}} 2p_i \begin{pmatrix} 1 \\ \mathbf{c}_i \end{pmatrix} = 2\gamma L_{\mathbf{b}} \begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 2 \\ \mathbf{0} \end{pmatrix}$ so we have a valid POVM.

6 \implies 3: Immediate from form of scaled down \mathcal{E}_A and \mathbf{a} in preliminaries.

3 \implies 6: If \mathbf{a} is on the scaled down surface then $\mathbf{a}' + T' \mathbf{x}' = \mathbf{a}' + T' \mathbf{b}$ where $x' = b$ and $\mathbf{x}' \in \ker(T')^\perp$. Hence $\mathbf{x}' - \mathbf{b} \in \ker(T')$. The only way the difference between two vectors of the same length can be perpendicular to one of them is if they are the same, and so 6 follows.

4 \implies 5: Suppose $\sum q_i \mathbf{x}_i = \mathbf{0}$, $\sum q_i = 1$ with $\mathbf{x}_i \in \mathcal{E}_B$. Recalling that swapping parties sends $\Theta \rightarrow \Theta^T$ we see that there exists Y_i in the forward light-cone with $\begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix} = \frac{1}{2}\Theta^T Y_i$. But then $\Theta^T \frac{1}{2} \sum_i q_i Y_i = \sum_i q_i \frac{1}{2}\Theta^T Y_i = \sum_i q_i \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$.

5 \implies 4: Suppose there exists Y with $\frac{1}{2}\Theta^T Y = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$. If $Y = \begin{pmatrix} t \\ \mathbf{y} \end{pmatrix}$ is in the forward light cone (i.e. $t \geq y$) then \mathcal{E}_B itself contains the maximally mixed state and we are done. Otherwise, notice that $Y_1 = \begin{pmatrix} y-t \\ 0 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} y \\ \mathbf{y} \end{pmatrix}$ are in the forward light cone. Writing $\frac{1}{2}\Theta^T Y_i$ as $q_i \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix}$ we have $\mathbf{x}_i \in \mathcal{E}_B$. Noting that $Y = \sum_i Y_i$ we have $\sum_i \frac{1}{2}\Theta^T Y_i = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$, in other words $\sum_i q_i = 1$ and $\sum_i q_i \mathbf{x}_i = \mathbf{0}$.

5 \iff 6: $\text{range}(A^T) = \ker(A)^\perp$ is a theorem of linear algebra, which follows straightforwardly from the singular value decomposition. \square

Appendix C: Reconstructing the state from its steering ellipsoid and the Bloch vectors

Alice's steering ellipsoid \mathcal{E}_A is described by matrix Q_A and centre \mathbf{c}_A . One quantum state that corresponds to it is the "canonical" state $\tilde{\rho}$ with Bloch vectors $\mathbf{b} = \mathbf{0}$, $\tilde{\mathbf{a}} = \mathbf{c}_A$ and T-matrix $\tilde{T} = \sqrt{Q_A}O$, with $O \in O(3)$ (since $Q_A = \tilde{T}\tilde{T}^T$). Written explicitly it is

$$\tilde{\rho} = \frac{1}{4}(\mathbb{1} + \mathbf{c}_A \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \sum_{i,j=1}^3 (\sqrt{Q_A}O)_{ij} \sigma_i \otimes \sigma_j) \quad (\text{C1})$$

The orthogonal matrix O is not fixed.

If in addition to Q_A, \mathbf{c}_A we are also given the Bloch vectors \mathbf{a}, \mathbf{b} , a corresponding state is

$$\rho = \mathbb{1} \otimes \sqrt{2\rho_B} \tilde{\rho} \mathbb{1} \otimes \sqrt{2\rho_B} \quad (\text{C2})$$

where

$$\rho_B = \frac{1}{2}(\mathbb{1} + \mathbf{b} \cdot \boldsymbol{\sigma}) \quad (\text{C3})$$

This is because $\sqrt{2\rho_B}$ is a SLOCC operator for $\mathbf{b} \neq \mathbf{0}$ and Alice's ellipsoid is invariant under such local operations on Bob's qubit.

Note that

$$\text{tr}(\rho \mathbb{1} \otimes \boldsymbol{\sigma}) = \mathbf{b}, \quad (\text{C4})$$

where $\boldsymbol{\sigma}$ is a vector of Paulis, as required. However

$$\text{tr}(\rho \boldsymbol{\sigma} \otimes \mathbb{1}) = \mathbf{c}_A + \sqrt{Q_A}O\mathbf{b} \quad (\text{C5})$$

and the right hand side of this should be \mathbf{a} . This constrains the orthogonal matrix O as a solution to

$$O\mathbf{b} = (\sqrt{Q_A})^{-1}(\mathbf{a} - \mathbf{c}_A) \quad (\text{C6})$$

We can split up the orthogonal matrix into two orthogonal rotations: $O = MB$ where $M \in \text{SO}(3)$ (properly) rotates \mathbf{b} to $(\sqrt{Q_A})^{-1}(\mathbf{b} - \mathbf{c}_A)$ about a unit vector perpendicular to the plane they lie in, and $B \in O(3)$ rotates about \mathbf{b} , that is $B\mathbf{b} = \mathbf{b}$. The equation (C6) only determines M .

Define the unitary matrix U_B as the one corresponding to B , it has the property $[U_B, \rho_B] = 0$. Then we may write

$$\tilde{\rho} = \mathbb{1} \otimes U_B \chi \mathbb{1} \otimes U_B^\dagger \quad (\text{C7})$$

where

$$\chi = \mathbb{1} \otimes U_B^\dagger \tilde{\rho} \mathbb{1} \otimes U_B \quad (\text{C8})$$

$$= \frac{1}{4}(\mathbb{1} + \mathbf{c}_A \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \sum_{i,j=1}^3 (\sqrt{Q_A}M)_{ij} \sigma_i \otimes \sigma_j) \quad (\text{C9})$$

Thus

$$\rho = \mathbb{1} \otimes \sqrt{2\rho_B} U_B \chi \mathbb{1} \otimes U_B^\dagger \sqrt{2\rho_B} \quad (\text{C10})$$

$$= \mathbb{1} \otimes U_B \sqrt{2\rho_B} \chi \mathbb{1} \otimes \sqrt{2\rho_B} U_B^\dagger \quad (\text{C11})$$

since if $[U_B, \rho_B] = 0$ then $[U_B, \sqrt{\rho_B}] = 0$.

Therefore the state ρ compatible with $Q_A, \mathbf{c}_A, \mathbf{a}, \mathbf{b}$ is

$$\rho = \frac{1}{4} \left(\mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} + \sum_{i,j=1}^3 T_{ij} \sigma_i \otimes \sigma_j \right) \quad (\text{C12})$$

where $T = RB$ and

$$\begin{aligned} R_{ij} &= \text{tr}(\mathbb{1} \otimes \sqrt{2\rho_B} \chi \mathbb{1} \otimes \sqrt{2\rho_B} \sigma_i \otimes \sigma_j) \quad (\text{C13}) \\ &= (c_A)_i b_j + \sum_{k=1}^3 (\sqrt{Q_A M})_{ik} \text{tr}(\sqrt{\rho_B} \sigma_k \sqrt{\rho_B} \sigma_j) \quad (\text{C14}) \end{aligned}$$

and so we can reconstruct the state up to a local unitary B on Bob's qubit that leaves it invariant.

Appendix D: Steering ellipsoids in a tetrahedron in the Bloch sphere correspond to separable states

We prove that for any \mathcal{E} inside a tetrahedron inside the Bloch sphere, there is a separable state with $\rho_B = \frac{1}{2}\mathbb{1}$ and \mathcal{E} as its steering ellipse. We present the proofs for each possible dimension of \mathcal{E} separately, although each one is basically a slightly more involved version of the previous one. Note that in the 0 and 1 dimensional cases the requirement to fit inside a tetrahedron is trivially satisfied by any \mathcal{E} inside the Bloch sphere.

This result suffices to show that any state with an ellipse that fits inside a tetrahedron is separable by the following argument. Suppose ρ_{AB} has an ellipse that fits inside the tetrahedron. If $b = 1$ then ρ_{AB} is a product state and we are done. Otherwise, apply a SLOCC operator to Bob and obtain $\tilde{\rho}_{AB}$ with $\mathbf{b} = 0$, recalling that SLOCC operators cannot change a state from being entangled to separable. This will leave Alice's ellipse unchanged whilst moving her reduced state to the centre of her ellipse. Since by the above statement there exists a separable state with the correct ellipse and reduced states, $\tilde{\rho}_{AB}$ must equal the separable state up to a choice of basis for Bob, and hence must itself be separable.

In fact the separable states constructed below use a number of product states equal to the dimension of the ellipse plus one. Since the SLOCC operator and choice of basis for Bob do not affect the number of product states in a decomposition, we furthermore have that ρ_{AB} can be built using that number of product states.

1. 0-dimensional

If the steering ellipsoid is a single point \mathbf{r} then simply take ρ with Bloch vector \mathbf{r} and let $\rho_{AB} = \rho \otimes \frac{1}{2}\mathbb{1}$.

2. 1-dimensional

Suppose \mathcal{E} is a line segment from \mathbf{r}_0 to \mathbf{r}_1 . Take ρ_i with Bloch vectors \mathbf{r}_i and let $\rho_{AB} = \frac{1}{2} \sum_i \rho_i \otimes |i\rangle\langle i|$.

3. 2-dimensional

If an ellipse fits inside a tetrahedron in the Bloch sphere, it also fits inside a triangle in the Bloch sphere [15]. Therefore, suppose an ellipse \mathcal{E} fits within a triangle in the Bloch sphere whose vertices are $\{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2\}$. Without loss of generality we can take the ellipse to be tangent to each edge of the triangle, at points $\{\mathbf{s}_i\}$ where \mathbf{s}_i is on the face opposite to \mathbf{r}_i . Denote the centre of the ellipse by \mathbf{c} . Clearly there exists unique $p_i \geq 0$ such that $\sum_i p_i \mathbf{r}_i = \mathbf{c}$ and $\sum_i p_i = 1$.

By the definition of an ellipse, there is an invertible affine transformation \mathcal{A} that maps \mathcal{E} to the unit circle in the (x, z) -plane, centred at the origin. Let ρ_i have Bloch vectors \mathbf{r}_i and $|\psi_i\rangle$ be such that the Bloch vector of $|\psi_i\rangle\langle\psi_i|$ is $-\mathcal{A}(\mathbf{s}_i)$. We claim that the (manifestly separable) state

$$\rho_{AB} = \sum_i p_i \rho_i \otimes |\psi_i\rangle\langle\psi_i| \quad (\text{D1})$$

has $\rho_B = \frac{1}{2}\mathbb{1}$ and that Alice's steering ellipsoid for this state is \mathcal{E} . To prove the first part, notice that the Bloch vector of ρ_B is $-\sum_i p_i \mathcal{A}(\mathbf{s}_i)$. Since \mathcal{A} is affine, the unit circle will be tangent to the triangle with vertices $\{\mathcal{A}(\mathbf{r}_i)\}$ at the points $\{\mathcal{A}(\mathbf{s}_i)\}$, and $\sum_i p_i \mathcal{A}(\mathbf{r}_i) = \mathcal{A}(\mathbf{c}) = \mathbf{0}$. Hence it suffices to prove

Lemma 1. *Suppose the triangle with vertices $\{\mathbf{v}_i\}$ contains the unit circle centered at the origin, and the circle is tangent to each edge of the triangle at the points $\{\mathbf{t}_i\}$ (where \mathbf{t}_i is on the edge opposite \mathbf{v}_i). Fix p_i by the requirements that $\sum_i p_i \mathbf{v}_i = \mathbf{0}$ and $\sum_i p_i = 1$. Then $\sum_i p_i \mathbf{t}_i = \mathbf{0}$.*

Proof. We use \mathbf{x} to represent points on or within the tetrahedron using normalized barycentric co-ordinates (x_0, x_1, x_2) where $\sum_i x_i = 1$ and $\vec{x} = \sum_i x_i \mathbf{v}_i$. Let A_0 be the volume of the triangle with vertices $\{\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2\}$, A_1 be the area of the triangle with vertices $\{\mathbf{v}_0, \mathbf{x}, \mathbf{v}_2\}$ and similarly for A_2 . Let A be the area of the original triangle (notice $A = \sum_i A_i$). Then $x_i = A_i/A$. By definition the barycentric co-ordinates of the origin are (p_0, p_1, p_2) .

Let L_i be the length of the edge opposite \mathbf{v}_i , and let $L = \sum_i L_i$. By using that the area of a triangle = $\frac{1}{2}$ (base) \times (perpendicular height) and noting that by the tangency assumption the relevant triangles have a perpendicular height of 1, we obtain that $p_i = L_i/L$.

Let $M_0^{(1)} = |\mathbf{v}_0 - \mathbf{t}_1|$, $M_0^{(2)} = |\mathbf{v}_0 - \mathbf{t}_2|$. In fact $M_0^{(1)} = M_0^{(2)}$ because they are both the unique length defined by the requirement of being from a fixed point to a point on the circle such that the line between them is tangent

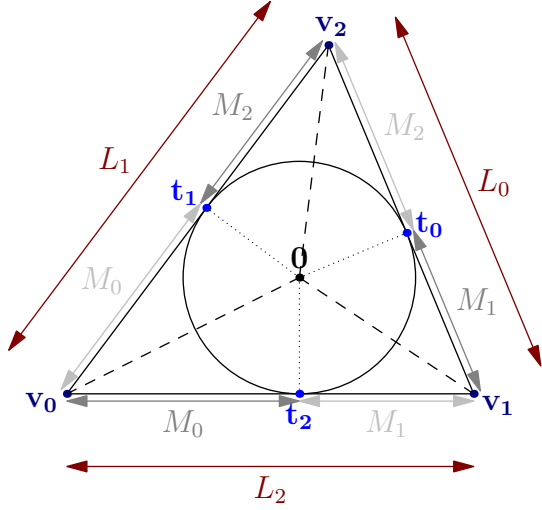


FIG. 3: The various quantities used in proving Lemma 1. The dashed lines form the three triangles used to show $p_i = L_i/L$, the dotted lines indicate their perpendicular heights (which are equal to the radius of the circle: 1).

to the sphere, so we can write this length simply as M_0 . Define the other two M_i by a similar argument. All this is illustrated in Figure 3. Notice that

$$L_0 = M_1 + M_2, \quad (D2)$$

$$L_1 = M_0 + M_2, \quad (D3)$$

$$L_2 = M_0 + M_1, \quad (D4)$$

$$(D5)$$

The barycentric co-ordinates of $\mathbf{t}_0, \mathbf{t}_1$ and \mathbf{t}_2 can now be calculated as

$$(0, M_2, M_1)/L_0, \quad (D6)$$

$$(M_2, 0, M_0)/L_1, \quad (D7)$$

and

$$(M_1, M_0, 0)/L_2, \quad (D8)$$

respectively. Using $p_i = L_i/L$ and the fact that barycentric coordinates respect convex combinations the required result is now immediate. \square

Suppose Bob projects qubit onto $|\psi\rangle$ and the orthogonal state. Since $\rho_B = \frac{1}{2}\mathbb{1}$ he will obtain each outcome with probability $\frac{1}{2}$. Therefore if he obtains the $|\psi\rangle$ outcome then Alice's state will be

$$\rho_A(|\psi\rangle) := \frac{\text{tr}_B(\rho_{AB}(I \otimes |\psi\rangle\langle\psi|))}{\text{tr}(\rho_B|\psi\rangle\langle\psi|)} \quad (D9)$$

$$= \frac{\sum_i p_i \rho_i |\langle\psi_i|\psi\rangle|^2}{\frac{1}{2}} \quad (D10)$$

$$= 2 \sum_i p_i \rho_i |\langle\psi_i|\psi\rangle|^2 \quad (D11)$$

Recalling that the Bloch vector of $|\psi_i\rangle\langle\psi_i|$ is $-\mathcal{A}(\mathbf{s}_i)$, then if $|\psi\rangle\langle\psi|$ has Bloch vector \mathbf{r} then the Bloch vector of $\rho_A(|\psi\rangle)$ will be

$$f(\mathbf{r}) := 2 \sum_i p_i \mathbf{r}_i \frac{1 - \mathbf{r} \cdot \mathcal{A}(\mathbf{s}_i)}{2} = \sum_i p_i \mathbf{r}_i (1 - \mathbf{r} \cdot \mathcal{A}(\mathbf{s}_i)). \quad (D12)$$

Let us extend this expression to all \mathbf{r} to define an affine function f . The statement that Alice's steering ellipsoid is \mathcal{E} is equivalent to the statement that \mathcal{E} is the image of the unit sphere under f . Since all the $\mathcal{A}(\mathbf{s}_i)$ are in the (x, z) -plane, we have $f((0, 1, 0)) = f(\mathbf{0})$, i.e. we can think of f as first projecting onto the (x, z) -plane and then applying some affine transformation. The image of the unit sphere under that projection is the unit disc, and so it suffices to check that \mathcal{E} is the image of the unit circle under f . Define $g(\mathbf{r}) = \mathcal{A}(f(\mathbf{r}))$. Since \mathcal{A} is invertible and maps \mathcal{E} to unit circle it suffices to prove that g is the identity on the (x, z) -plane. Since g is the composition of two affine functions it is also affine. By the definition of the p_i , $g(\mathbf{0}) = \mathcal{A}(\mathbf{c}) = \mathbf{0}$ so g is in fact linear. Hence it suffices to check that $g(\mathbf{u}_j) = \mathbf{u}_j$ for some spanning set of vectors $\{\mathbf{u}_j\}$. Since the triangle cannot be degenerate, its vertex set $\{\mathbf{r}_j\}$ span some plane. Since \mathcal{A} is invertible, $\{\mathcal{A}(\mathbf{r}_j)\}$ must span the (x, z) plane. For $i \neq j$, $\{\mathbf{0}, \mathcal{A}(\mathbf{s}_i), \mathcal{A}(\mathbf{r}_j)\}$ form a right-angle triangle, and $|\mathcal{A}(\mathbf{s}_i)| = 1$. Therefore $\mathcal{A}(\mathbf{r}_j) \cdot \mathcal{A}(\mathbf{s}_i) = 1$ whenever $i \neq j$. But $\sum_i p_i (1 - \mathbf{r} \cdot \mathcal{A}(\mathbf{s}_i)) = \sum_i p_i - \mathbf{r} \cdot (\sum_i p_i \mathcal{A}(\mathbf{s}_i)) = 1 - \mathbf{r} \cdot \mathbf{0} = 1$ (the penultimate equality is from Lemma 1). Hence $p_i (1 - \mathcal{A}(\mathbf{r}_j) \cdot \mathcal{A}(\mathbf{s}_i)) = \delta_{ij}$ and we are done.

4. 3-dimensional

Suppose an ellipsoid \mathcal{E} fits within a tetrahedron in the Bloch sphere whose vertices are $\{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$. Without loss of generality we can take the ellipsoid to be tangent to each face of the tetrahedron, at points $\{\mathbf{s}_i\}$ where \mathbf{s}_i is on the face opposite to \mathbf{r}_i . Denote the centre of the ellipsoid by \mathbf{c} . Clearly there exists unique $p_i \geq 0$ such that $\sum_i p_i \mathbf{r}_i = \mathbf{c}$ and $\sum_i p_i = 1$.

By the definition of an ellipsoid, there is an invertible affine transformation \mathcal{A} that maps \mathcal{E} to the unit sphere centred at the origin. Let ρ_i have Bloch vectors \mathbf{r}_i and $|\psi_i\rangle$ be such that the Bloch vector of $|\psi_i\rangle\langle\psi_i|$ is $-\mathcal{A}(\mathbf{s}_i)$. We claim that the (manifestly separable) state

$$\rho_{AB} = \sum_i p_i \rho_i \otimes |\psi_i\rangle\langle\psi_i| \quad (D13)$$

has $\rho_B = \frac{1}{2}\mathbb{1}$ and that Alice's steering ellipsoid for this state is \mathcal{E} . To prove the first part, notice that the Bloch vector of ρ_B is $-\sum_i p_i \mathcal{A}(\mathbf{s}_i)$. Since \mathcal{A} is affine, the unit sphere will be tangent to the tetrahedron with vertices $\{\mathcal{A}(\mathbf{r}_i)\}$ at the points $\{\mathcal{A}(\mathbf{s}_i)\}$, and $\sum_i p_i \mathcal{A}(\mathbf{r}_i) = \mathcal{A}(\mathbf{c}) = \mathbf{0}$. Hence it suffices to prove the following 3-dimensional analogue to Lemma 1:

Lemma 2. Suppose the tetrahedron with vertices $\{\mathbf{v}_i\}$ contains the unit sphere centered at the origin, and the

sphere is tangent each face of the tetrahedron at the points $\{\mathbf{t}_i\}$ (where \mathbf{t}_i is on the face opposite \mathbf{v}_i). Fix p_i by the requirements that $\sum_i p_i \mathbf{v}_i = \mathbf{0}$ and $\sum_i p_i = 1$. Then $\sum_i p_i \mathbf{t}_i = \mathbf{0}$.

Proof. We use \mathbf{x} to represent points on or within the tetrahedron using normalized barycentric co-ordinates (x_0, x_1, x_2, x_3) where $\sum_i x_i = 1$ and $\vec{x} = \sum_i x_i \mathbf{v}_i$. Let V_0 be the volume of the tetrahedron with vertices $\{\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, V_1 be the volume of the tetrahedron with vertices $\{\mathbf{v}_0, \mathbf{x}, \mathbf{v}_2, \mathbf{v}_3\}$ and so on. Let V be the volume of the original tetrahedron (notice $V = \sum_i V_i$). Then $x_i = V_i/V$. By definition the barycentric co-ordinates of the origin are (p_0, p_1, p_2, p_3) .

Let A_i be the area of the face opposite \mathbf{v}_i , and let $A = \sum_i A_i$. By using that the volume of a tetrahedron = $\frac{1}{3}$ (area of base) \times (perpendicular height) and noting that by the tangency assumption the relevant tetrahedra have a perpendicular height of 1, we obtain that $p_i = A_i/A$.

Let $A_{23}^{(0)}$ be the area of the triangle with vertices $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{t}_0\}$. Let $A_{23}^{(1)}$ be the area of the triangle with vertices $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{t}_1\}$. Now we have that $|\mathbf{v}_2 - \mathbf{t}_0| = |\mathbf{v}_2 - \mathbf{t}_1|$ because they are both the unique length defined by the requirement of being from a fixed point to a point on the sphere such that the line between them is tangent to the sphere. Similarly $|\mathbf{v}_3 - \mathbf{t}_0| = |\mathbf{v}_3 - \mathbf{t}_1|$. Hence the two triangles are congruent and we can simply write their areas as A_{23} . Define the other five A_{ij} by a similar argument. Notice that

$$A_0 = A_{12} + A_{13} + A_{23}, \quad (\text{D14})$$

$$A_1 = A_{02} + A_{03} + A_{23}, \quad (\text{D15})$$

$$A_2 = A_{01} + A_{03} + A_{13}, \quad (\text{D16})$$

$$A_3 = A_{01} + A_{02} + A_{12}. \quad (\text{D17})$$

The barycentric co-ordinates of $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2$ and \mathbf{t}_3 can now be calculated as

$$(0, A_{23}, A_{13}, A_{12})/A_0, \quad (\text{D18})$$

$$(A_{23}, 0, A_{03}, A_{02})/A_1, \quad (\text{D19})$$

$$(A_{13}, A_{03}, 0, A_{01})/A_2, \quad (\text{D20})$$

and

$$(A_{12}, A_{02}, A_{01}, 0)/A_3 \quad (\text{D21})$$

respectively. Using $p_i = A_i/A$ and the fact that barycentric coordinates respect convex combinations the required result is now immediate. \square

As in the 2-dimensional case we find that if Bob projects onto the state with Bloch vector \mathbf{r} than Alice's Bloch vector is

$$f(\mathbf{r}) := \sum_i p_i \mathbf{r}_i (1 - \mathbf{r} \cdot \mathcal{A}(\mathbf{s}_i)). \quad (\text{D22})$$

Let us extend this expression to all \mathbf{r} to define an affine function f . The statement that Alice's steering ellipsoid is \mathcal{E} is equivalent to the statement that \mathcal{E} is the image of the unit sphere under f . Define $g(\mathbf{r}) = \mathcal{A}(f(\mathbf{r}))$. Since \mathcal{A} is invertible and maps \mathcal{E} to unit sphere it suffices to prove that g is the identity. Since g is the composition of two affine functions it is also affine. By the definition of the p_i , $g(\mathbf{0}) = \mathcal{A}(\mathbf{c}) = \mathbf{0}$ so g is in fact linear. Hence it suffices to check that $g(\mathbf{u}_j) = \mathbf{u}_j$ for some spanning set of vectors $\{\mathbf{u}_j\}$. Since the tetrahedron cannot be degenerate, its vertex set $\{\mathbf{r}_j\}$ must be spanning. Since \mathcal{A} is invertible, $\{\mathcal{A}(\mathbf{r}_j)\}$ is also spanning. As in the 2-dimensional case, for $i \neq j$, $\{\mathbf{0}, \mathcal{A}(\mathbf{s}_i), \mathcal{A}(\mathbf{r}_j)\}$ form a right-angle triangle, and $|\mathcal{A}(\mathbf{s}_i)| = 1$. Therefore $\mathcal{A}(\mathbf{r}_j) \cdot \mathcal{A}(\mathbf{s}_i) = 1$ whenever $i \neq j$. But $\sum_i p_i (1 - \mathbf{r} \cdot \mathcal{A}(\mathbf{s}_i)) = \sum_i p_i - \mathbf{r} \cdot (\sum_i p_i \mathcal{A}(\mathbf{s}_i)) = 1 - \mathbf{r} \cdot \mathbf{0} = 1$ (the penultimate equality is from Lemma 2). Hence $p_i (1 - \mathcal{A}(\mathbf{r}_j) \cdot \mathcal{A}(\mathbf{s}_i)) = \delta_{ij}$ and we are done.

Appendix E: Discord and steering ellipsoids

Below we outline the condition for zero discord for Alice from either her or Bob's ellipsoid.

A state has zero discord for Alice iff her ellipsoid is a segment of a diameter.

The “only if” part: A general zero discord state for Alice $\rho = p|e\rangle\langle e| \otimes \rho_0 + (1-p)|\bar{e}\rangle\langle \bar{e}| \otimes \rho_1$ has $\langle e|\bar{e}\rangle = 0$ and

$$\mathbf{a} = t\mathbf{e} \quad (\text{E1})$$

$$\mathbf{b} = \mathbf{x} \quad (\text{E2})$$

$$T = \mathbf{e}\mathbf{y}^T \quad (\text{E3})$$

where $t = 2p - 1$, $\mathbf{e} = \langle e|\boldsymbol{\sigma}|e\rangle$ and $\mathbf{x} = \text{tr}[(p\rho_0 + (1-p)\rho_1)\boldsymbol{\sigma}]$, $\mathbf{y} = \text{tr}[(p\rho_0 - (1-p)\rho_1)\boldsymbol{\sigma}]$ [20].

Alice's steering ellipsoid \mathcal{E}_A has centre $\mathbf{c}_A = \left(\frac{t-\mathbf{x}\cdot\mathbf{y}}{1-x^2}\right)\mathbf{e}$ and matrix $Q_A = s_A^2 \mathbf{e}\mathbf{e}^T$ with

$$s_A^2 = \frac{1}{1-x^2} \left[(\mathbf{y} - t\mathbf{x})^T \left(\mathbb{1} + \frac{\mathbf{x}\mathbf{x}^T}{1-x^2} \right) (\mathbf{y} - t\mathbf{x}) \right] \quad (\text{E4})$$

So \mathcal{E}_A is a segment of the diameter.

The “if” part: suppose we are given \mathcal{E}_A , a segment of the diameter. Denote the states endpoints of the ellipsoid ρ_0 and ρ_1 . Alice's state can always be decomposed as $\rho_A = q\rho_0 + (1-q)\rho_1$. However since all the Bloch vectors of ρ_A, ρ_0, ρ_1 are collinear they will eigendecompose into the same pair of orthogonal states, call them $|\psi\rangle, |\bar{\psi}\rangle$. Writing $\rho_i = p_i|\psi\rangle\langle\psi| + (1-p_i)|\bar{\psi}\rangle\langle\bar{\psi}|$, for $i = 0, 1$ then $\rho_A = p|\psi\rangle\langle\psi| + (1-p)|\bar{\psi}\rangle\langle\bar{\psi}|$ with $p = qp_0 + (1-q)p_1$. Then the joint state

$$\rho = p|\psi\rangle\langle\psi| \otimes \beta_0 + (1-p)|\bar{\psi}\rangle\langle\bar{\psi}| \otimes \beta_1 \quad (\text{E5})$$

is a zero discord state for Alice with the correct \mathcal{E}_A and ρ_A for any mixed states on Bob's side β_0, β_1 .

There is zero discord for Alice iff Bob's ellipsoid is a line segment and the length of Alice's Bloch vector is equal to the distance from the centre of Bob's ellipsoid to his Bloch vector divided by the radius of his ellipsoid.

The “only if” part can easily be checked: it requires $a = \frac{|c_B - b|}{s_B}$. Since, after some algebra,

$$c_B = \frac{\mathbf{x} - t\mathbf{y}}{1 - t^2} \quad (\text{E6})$$

$$Q_B = \frac{1}{(1 - t^2)^2} (\mathbf{y} - t\mathbf{x})(\mathbf{y} - t\mathbf{x})^T \quad (\text{E7})$$

then $|c_B - b| = \frac{t|\mathbf{y} - t\mathbf{x}|}{1 - t^2} = as_B$.

For the “if” part, let ρ_0 and ρ_1 be the endpoints of Bob's ellipsoid, and let the state corresponding to Alice's Bloch vector have eigen-decomposition $\rho_A = p_0|\psi_0\rangle\langle\psi_0| + p_1|\psi_1\rangle\langle\psi_1|$. Then the joint state

$$\rho = p_0|\psi_0\rangle\langle\psi_0| \otimes \rho_0 + p_1|\psi_1\rangle\langle\psi_1| \otimes \rho_1 \quad (\text{E8})$$

has zero discord for Alice, the correct Bloch vector for Alice and the correct ellipsoid for Bob. If necessary swapping ρ_0 and ρ_1 , it also has the right Bloch vector for Bob. Bob's ellipsoid \mathcal{E}_B is invariant under local unitaries on Alice's qubit, so Alice and Bob's actual state is therefore equivalent to ρ up to this transformation, which preserves discord.

Appendix F: Volume formula for the steering ellipsoid

The volume of any ellipsoid is proportional to the product of its eigenvalues $V = \frac{4\pi}{3} s_1 s_2 s_3$. Therefore the \mathcal{E}_A has volume $V_A = \frac{4\pi}{3} |\sqrt{\det Q_A}|$, which may be rewritten as

$$V_A = \frac{4\pi}{3} \frac{|\det(T - \mathbf{a}b^T)|}{(1 - b^2)^2} = \frac{4\pi}{3} \frac{|\det\Theta|}{(1 - b^2)^2}. \quad (\text{F1})$$

To express this in terms of the density matrix ρ , we use the equation $\Theta = 2\Upsilon\rho^R\Upsilon^T$ [9], where the unitary matrix Υ is given in equation (A2) and R denotes a reshuffling operation: if $\rho = \sum_{i,j=0}^1 \rho_{ij;kl} |ij\rangle\langle kl|$ then $\rho^R = \sum_{i,j=0}^1 \rho_{ik;jl} |ij\rangle\langle kl|$. We also require a curious relation that holds for any 4×4 or 9×9 matrix M . For any such M we have that

$$\det M = \det M^{T_B} - \det(M^{T_B})^R. \quad (\text{F2})$$

Applying this relation to (F1), together with the reshuffled form of Θ , we obtain

$$V_A = \frac{64\pi}{3} \frac{|\det \rho - \det \rho^{T_B}|}{(1 - b^2)^2}. \quad (\text{F3})$$

Appendix G: The steering ellipsoid zoo

In this section we illustrate the main types of ellipsoid.

1. Entangled states

For every pure entangled state the ellipsoid coincides with the Bloch sphere. When the state is mixed and entangled, the ellipsoid does not satisfy the tetrahedral condition because, loosely speaking, the ellipsoid is either *too big* (with volume $V \geq V_\star > 0$) or *too near* (large c) to the surface of the Bloch sphere, see figure 4. Every entangled state is completely steerable.

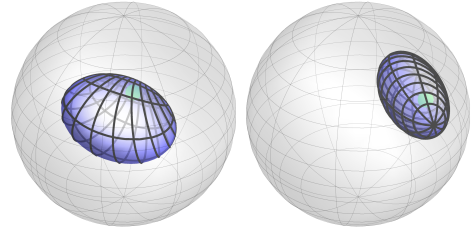


FIG. 4: A generic entangled state ρ_{AB} : both ellipsoids \mathcal{E}_A and \mathcal{E}_B are always full rank and neither can be inscribed within a tetrahedron within the Bloch sphere.

2. Separable states with full-dimensional ellipsoids

Separable states admit a convex decomposition in terms of product states, and have “more classical” correlations. Steering is still possible, however the steering ellipsoids necessarily obey the tetrahedral condition, as in figure 5.

If the state has a three dimensional \mathcal{E}_A then it has non-zero obesity and non-zero discord, and furthermore, it can be written as a mixture of just four product states. Such states are also completely steerable.

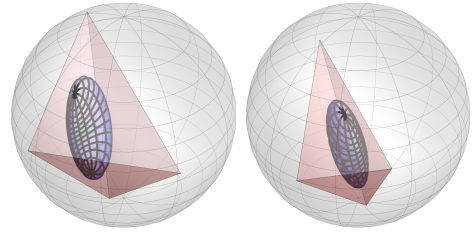


FIG. 5: A generic separable state ρ_{AB} , where both ellipsoids \mathcal{E}_A and \mathcal{E}_B are full rank fit inside a tetrahedron.

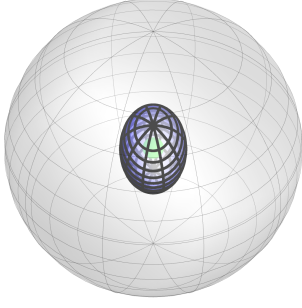


FIG. 6: A Bell-diagonal state: the ellipsoid is centred at the origin and its semiaxes are given by the three singular values of T . The vector of these singular values $\mathbf{t} = (t_1, t_2, t_3)$ lives in a tetrahedron with vertices at $(1,1,-1), (1,-1,1), (-1,1,1), (-1,-1,-1)$, and when it is inside an octahedron inside of this tetrahedron, then the state is necessarily separable. This defines the set of ellipsoids that fit inside the nested tetrahedron (these are not the same tetrahedra).

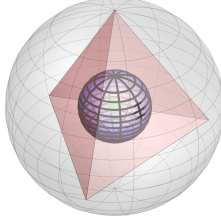


FIG. 7: A Werner state. The steering ellipsoid \mathcal{E}_A is a sphere centered at the origin. The ellipsoid fits inside a tetrahedron when its radius is less than $\frac{1}{3}$ and thus the state is separable.

3. Steering pancakes

The set of states that Bob can steer Alice to may become degenerate, and form a *two-dimensional* set. This “steering pancake” will not only fit inside a tetrahedron, but will fit within a *triangle* that is inscribed within the Bloch sphere as shown in 8. Recall that we have a novel

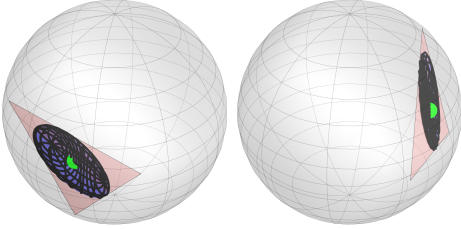


FIG. 8: A generic separable state ρ_{AB} , where both ellipsoids \mathcal{E}_A and \mathcal{E}_B are steering pancakes.

feature for some steering pancakes (and steering needles) of *incomplete* steering. For steering pancakes we have complete steering of qubit A if and only if the affine span of \mathcal{E}_B contains the origin of the Bloch sphere.

4. Steering needles

The steering can become even more degenerate, and the steering set collapse to a one-dimensional line segment, or “steering needle”. These states include perfectly classical (doubly zero-discord states) with needles being radial (figure 9), but also includes non-zero discord states for which either one or both of the steering needles \mathcal{E}_A and \mathcal{E}_B is not radial (figure 10). Being radial is indicated by dashed lines on the figures, which depict diameters.

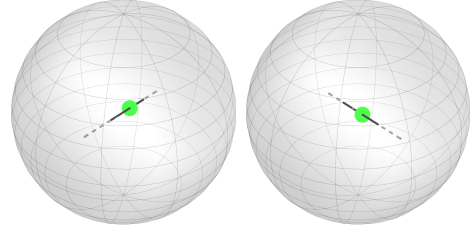


FIG. 9: A doubly zero discord state, where both \mathcal{E}_A and \mathcal{E}_B are radial line segments.

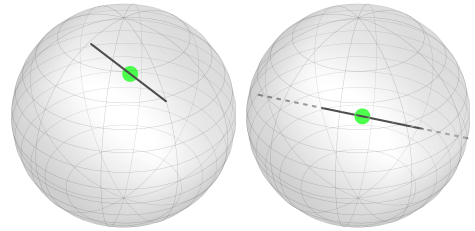


FIG. 10: A state with zero discord at B , but *non-zero* discord at A . We have \mathcal{E}_B being a radial line segment, but \mathcal{E}_A is not a radial line segment.

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